2026 ANDRE TUTORING SAMPLE PAPER

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- Write your Centre Number and Student Number at the top of this page

Total marks: 70

Section I – 10 marks (pages 2–7)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 9–36)

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–20.

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1 What is the solution to the inequality $\dfrac{1}{x-4} \geq \dfrac{1}{x+2}$

A. $-2 < x < 4$

B. $x \leq -2$ or $x \geq 4$

C. $x < -2$ or $x > 4$

D. $-\infty < x < -2$ and $4 < x < \infty$

2 Use the substitution $u = \sqrt{e^x - 1}$ to evaluate $\displaystyle\int_0^1 \sqrt{e^x - 1}\ dx$

A. $2\sqrt{e-1} - 2\tan^{-1}\left(\sqrt{e-1}\right)$

B. $2\sqrt{e-1} + 2\tan^{-1}\left(\sqrt{e-1}\right)$

C. $\sqrt{e-1} - \tan^{-1}\left(\sqrt{e-1}\right)$

D. $\ln(e-1)$

3 Two vectors are shown.

$$\begin{pmatrix} x \\ 1-x \end{pmatrix} \text{ and } \begin{pmatrix} x^3 \\ x \end{pmatrix}$$

The dot product of the vectors is maximised when x is equal to?

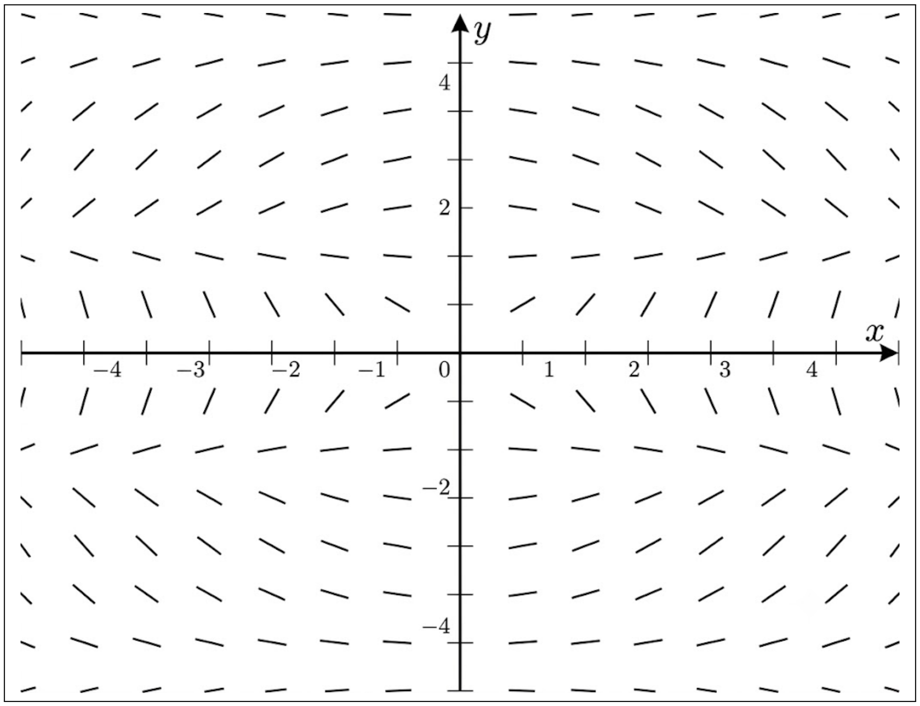
A. -1

B. 0

C. $\dfrac{1}{2}$

D. 1

4 A slope field is shown.



Which of the following differential equations matches the slope field?

A. $\dfrac{dy}{dx} = \dfrac{x\cos y}{y}$

B. $\dfrac{dy}{dx} = \dfrac{y\sin x}{x}$

C. $\dfrac{dy}{dx} = x\cos y$

D. $\dfrac{dy}{dx} = \dfrac{xy}{\cos y}$

5 A target archer hits the bullseye with a probability of p for each shot. Suppose the archer takes 50 independent shots. Which of the following expressions represents the probability that the archer hits the bullseye exactly 32 times OR exactly 33 times?

A. $p^{33}(1-p)^{18}\left[\binom{50}{32}+\binom{50}{33}\right]$

B. $p^{32}(1-p)^{17}\left[\binom{50}{32}(1-p)+\binom{50}{33}p\right]$

C. $\binom{51}{33}p^{32}(1-p)^{17}$

D. $\binom{51}{33}p^{33}(1-p)^{18}$

6 Let $P(x)=x^4+ax^3+15x^2+cx+24$ be a quartic polynomial with roots $\alpha,\alpha,\beta,\gamma$. It is known that $\beta+\gamma=1$ and $\beta\gamma=6$.

If the roots $\alpha,\alpha,\beta,\gamma$ are real, find the value of the coefficient a.

A. -5

B. -3

C. 3

D. 5

7 How many solutions are there to the equation $\sin x+\sin 2x+\sin 3x=1+\cos x+\cos 2x$ in the interval $0\le x\le 4\pi$?

A. 8

B. 10

C. 12

D. 14

8 Let the following definite integrals be defined:

(I) $\int_0^{\pi} \sin^4 x\,dx$ (II) $\int_0^{\pi} \sin^2 x \cos^2 x\,dx$ (III) $\int_0^{\pi} (\sin x + \cos x)^2 (\sin x - \cos x)^2\,dx$

Which of the following identifies the relative magnitudes of these integrals?

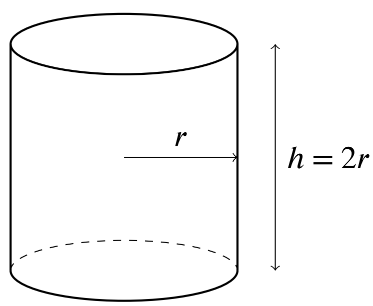
A. (III) > (I) > (II)

B. (I) > (III) > (II)

C. (II) > (III) > (I)

D. (I) > (II) > (III)

9 A closed cylindrical canister is being manufactured such that its height h is always twice its radius r ($h = 2r$). If the volume of the cylinder is increasing at a rate of 16π cm^3/s, find the rate at which the surface area S is increasing at the instant when the radius $r = 4$ cm.



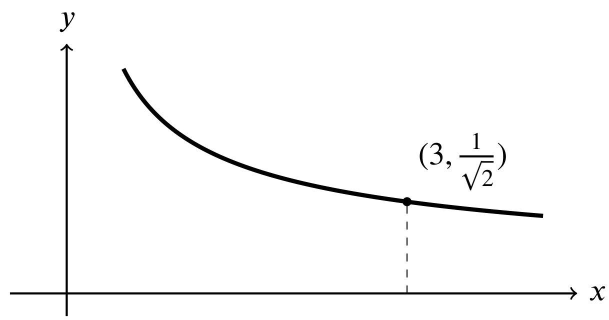
A. 4π cm^2/s

B. 8π cm^2/s

C. 12π cm^2/s

D. 16π cm^2/s

10 The graph below shows the function $y = \frac{1}{\sqrt{f(x)}}$. Given that the tangent to the curve at $x = 3$ has a gradient of $-\frac{1}{4\sqrt{2}}$, determine the value of $\frac{d}{dx}(xf(x))$ when $x = 3$.



A. 4

B. 5

C. 6

D. 7

Section II

60 marks
Attempt Questions 11-14
Allow about 1 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet

(a) Sketch the graph of the function $f(x) = \dfrac{3}{\sin^{-1}(2x+1)}$ **2**

(b) The roots of $P(x) = x^3 - \alpha x^2 + \beta x - 2\gamma$ are α, β, and γ. **3**

What is the value of $\alpha^2 + \beta^2 + \gamma^2$?

(c) Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin^2(x)}{\cos^4(x)}\,dx$ by using the substitution $u = \tan(x)$. **3**

(d) Prove by mathematical induction that for all integers $n \geq 1$: **4**

$$\underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}_{n \text{ radicals}} = 2\cos\left(\frac{\pi}{2^{n+1}}\right)$$

(e) There is a box with 4 red balls, 4 blue balls and 4 green balls in it. Balls of the same colour are identical. Calculate the number of distinct ways to place the balls in a line such that no two red balls are adjacent. **3**

Question 12 (15 marks) Use the Question 12 Writing Booklet

(a) The population of an insect colony $P(t)$ satisfies the differential equation below, where P is the population, and t is the number of days which has elapsed.

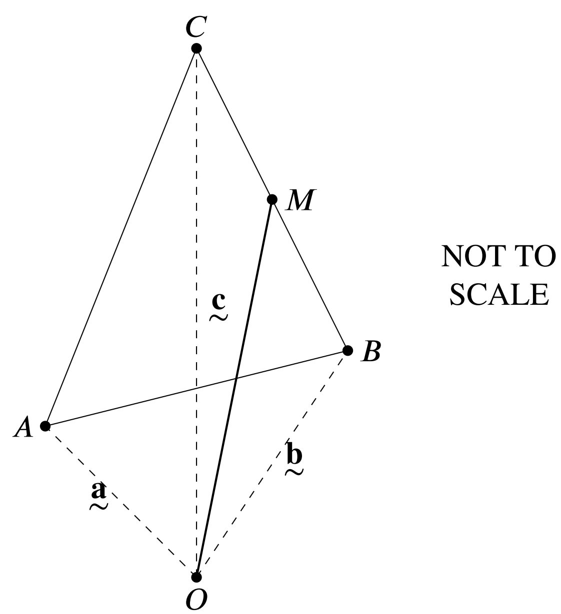
$$\frac{dP}{dt} = \frac{P}{t} + P^2 t$$

(i) Show that the substitution $v = \frac{P}{t}$ transforms the equation into: **2**

$$\frac{dv}{dt} = v^2 t^2$$

(ii) Solve the differential equation for $P(t)$ given that after 1 day there are 200 insects in the colony, and hence calculate the population after 2 days. **2**

(b) Consider points A, B, and C in 3 dimensions denoted by the vectors $\underset{\sim}{\mathbf{a}}, \underset{\sim}{\mathbf{b}}$, and $\underset{\sim}{\mathbf{c}}$ respectively. Let O be the origin. Let M be the midpoint of BC. Prove that if $\triangle OBC$ is isosceles and $\underset{\sim}{\mathbf{a}} \cdot (\underset{\sim}{\mathbf{b}} - \underset{\sim}{\mathbf{c}}) = 0$, then AM is perpendicular to BC. **4**



Question 12 continues on page 10

Question 12 (continued)

(c) Circuit boards are produced in batches of size n, where the probability of a board being defect-free depends on the number of circuit boards produced by the following relation **4**

$$p = 1 - \frac{4}{n}.$$

A sample proportion $\hat{P}$ is taken. It is given that $P(\hat{P} - p > 0.1) \approx 0.025$. By using the approximation:

$$n^3 - 1600n + 6400 \approx (n-4)(n^2 - 1600)$$

Show that n approximately equals 40.

(d) In an experiment studying wave interference, the displacement of a combined signal is modelled by: **3**

$$D(x) = \cos(2x) - \sin(2x) \text{ for } 0 \leq x \leq 2\pi$$

Where x represents the phase angle in radians. By using an appropriate t-formulae substitution, determine the values of x for which the displacement $D(x) < -1$.

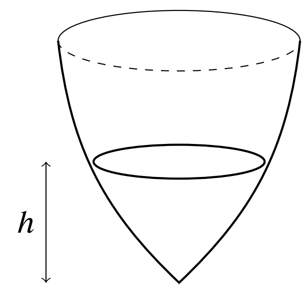
End of Question 12

Question 13 (16 marks) Use the Question 13 Writing Booklet

(a) By considering the expansion of $(1+x)^n$, show that: **3**

$$\sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k} = \frac{2^{n+1}-1}{n+1}$$

(b) A container is formed by rotating the curve $y = \sin^{-1}(x)$ about the y-axis for $0 \le y \le \frac{\pi}{2}$. Water is poured into the container at a constant rate of k m^3/min.



Container

(i) By considering the volume of revolution of the curve about the y-axis, show that the volume V of water in the container when the depth is h is given by: **2**

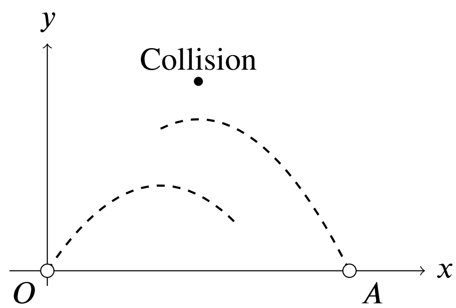
$$V = \frac{\pi}{2}\left(h - \frac{1}{2}\sin 2h\right)$$

(ii) If the water level is rising at a rate of $\frac{2}{\pi}$ m/min when the depth $h = \frac{\pi}{4}$ m, determine the rate k at which water is being poured into the container. **2**

Question 13 continues on page 10

Question 13 (continued)

(c) Two projectiles, P_1 and P_2, are launched simultaneously from $O(0,0)$ and $A(d,0)$ respectively. P_1 is launched with velocity $\mathbf{v}_1 = u_1 \cos\alpha \mathbf{i} + u_1 \sin\alpha \mathbf{j}$ and P_2 is launched with velocity $\mathbf{v}_2 = -u_2 \cos\beta \mathbf{i} + u_2 \sin\beta \mathbf{j}$. Both are subject to gravity g.



(i) If both projectiles collide at time $t = T$ at the same coordinate (x_c, y_c), show that the distance d between the launch points is given by: **2**

$$d = T(u_1 \cos\alpha + u_2 \cos\beta)$$

(ii) Given $u_1 = u_2 = 20$ m/s, $d = 40$ m, and $g = 10$ m/s^2 and $\alpha = \beta$. Find the required launch angles α such that they collide at the maximum possible height. **4**

(d) Consider the function $f(x) = \dfrac{x+a}{x-a}$ for $x \neq a$, where $a > 0$. Determine the value of a such that the equation: **3**

$$f(x) = f^{-1}\left(\frac{x}{a^2}\right)$$

Holds as an identity for all x in the domain.

End of Question 13

Question 14 (14 marks) Use the Question 14 Writing Booklet

(a) Let $\mathbf{a}$ and $\mathbf{b}$ be vectors such that $|\mathbf{a}| = 2$ and $|\mathbf{b}| = 3$. The magnitude of the resultant vector is given by $f(\theta) = |\mathbf{a} + \mathbf{b}|$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

(i) Show that $f(\theta) = \sqrt{13 + 12\cos\theta}$. **2**

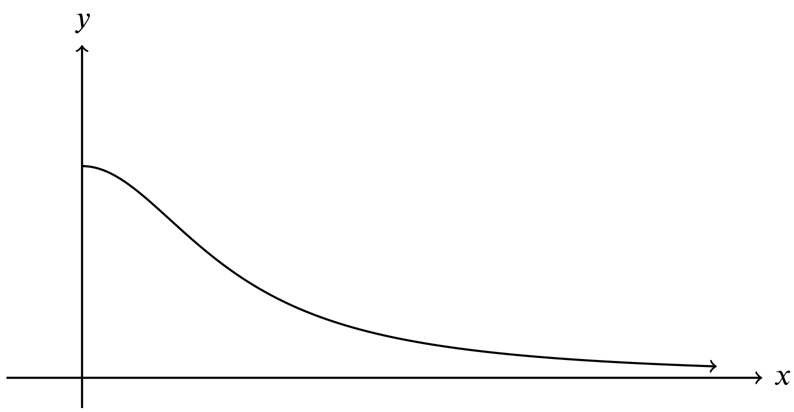
(ii) Given that the vectors are constrained such that the dot product $\mathbf{a} \cdot \mathbf{b} \leq 2$, find the maximum value of $f(\theta)$ and the value of θ at which it occurs. **2**

(b) Consider the function $f(x) = \tan^{-1}(x)$ for $x \geq 0$.

(i) Show that for $x > 0$, the following inequality holds: **2**

$$\tan^{-1}\left(\frac{1}{1+(x+1)^2}\right) < \tan^{-1}(x+1) - \tan^{-1}(x) < \tan^{-1}\left(\frac{1}{1+x^2}\right)$$

(ii) The graph of $g(x) = \dfrac{1}{1+x^2}$ is shown below: **3**



It is given that for $x > 0$

$$\frac{1}{1+(x+1)^2} < \tan^{-1}(x+1) - \tan^{-1}(x) < \frac{1}{1+x^2}$$

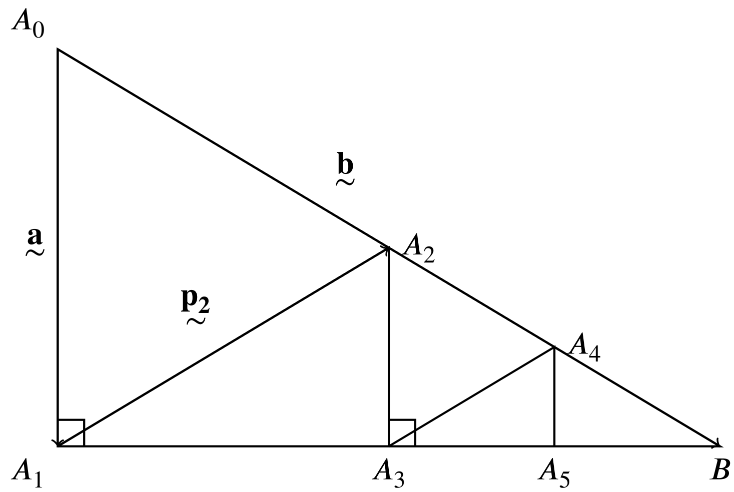
By considering the sum of rectangles of width 1 under $g(x)$, prove that:

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2} < \frac{\pi}{2}$$

Question 14 continues on page 10

(c) Consider the nested triangles where A_n is the midpoint of the line segment $A_{n-2}B$, for $n \geq 2$. Let $\overrightarrow{A_0A_1} = \underset{\sim}{\mathbf{a}}$ and $\overrightarrow{A_0B} = \underset{\sim}{\mathbf{b}}$. Let $\mathbf{p}_n$ be the vector between A_{n-1} and A_n. Prove by induction for $n \geq 1$ that: **5**

$$\underset{\sim}{\mathbf{p_n}} = \frac{1}{2^{n-1}}\left[\underset{\sim}{\mathbf{a}}\cos\left(\frac{(n-1)\pi}{2}\right) + \underset{\sim}{\mathbf{b}}\left(\frac{1-(-1)^{n-1}}{4}\right)\right]$$



End of paper

Mathematic Extension 1 Paper
Marking Guidelines

Section I

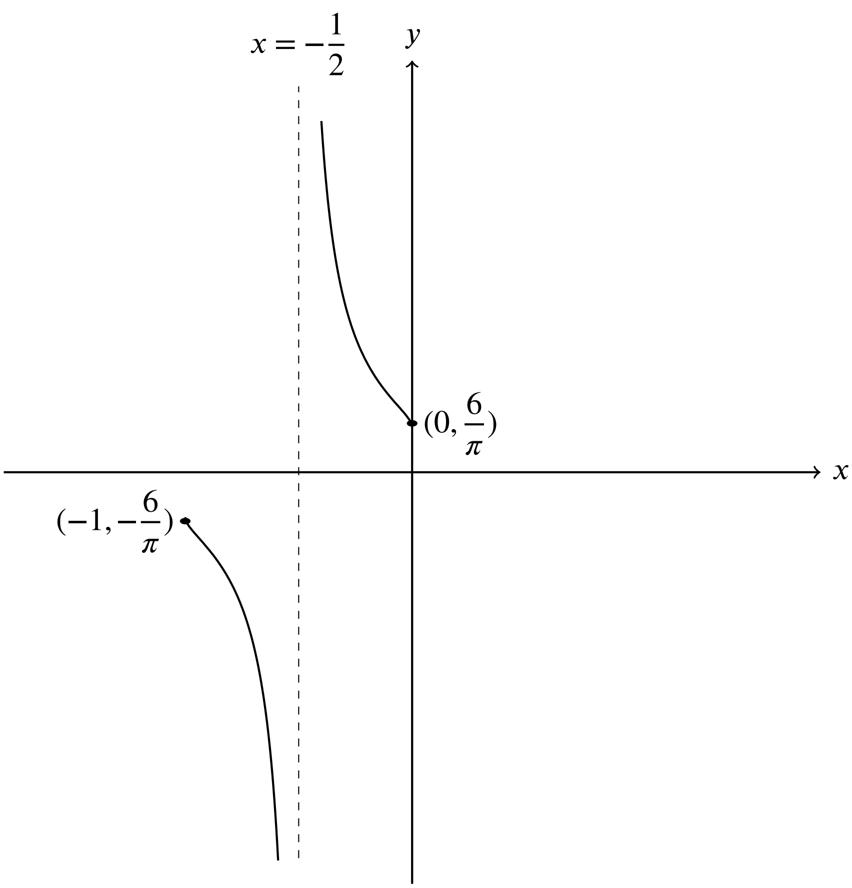
Multiple-choice Answer Key

Question	Answer
1	A
2	A
3	A
4	C
5	B
6	B
7	B
8	B
9	B
10	D

Question 11 (a)

Criteria	Marks
• Correctly sketches the graph indicating natural domain endpoints and asymptotes	2
• Determines correct domain boundaries or identifies the key vertical asymptote location	1

Sample answer:



Question 11 (b)

Criteria	Marks
• Provides the correct value for $\alpha^2+\beta^2+\gamma^2$	3
• Establishes correct numerical tracking relations for α,β,γ by solving simultaneous equations	2
• Relates polynomial coefficients to sum and product of roots via Vieta's relations	1

Sample answer:

$$\alpha+\beta+\gamma=\alpha$$

$$\beta+\gamma=0$$

$$\gamma=-\beta$$

$$\alpha\beta+\beta\gamma+\alpha\gamma=\beta$$

$$\alpha\beta-\beta^2-\alpha\beta=\beta$$

$$-\beta^2=\beta$$

$$\beta=-1$$

$$\gamma=1$$

$$\alpha\beta\gamma=2\gamma$$

$$\alpha(-1)(1)=2(1)$$

$$-\alpha=2$$

$$\alpha=-2$$

$$\begin{aligned}\alpha^2+\beta^2+\gamma^2&=(-2)^2+(-1)^2+(1)^2\\&=4+1+1\\&=6\end{aligned}$$

Question 11 (c)

Criteria	Marks
• Evaluates the exact numerical constant value of the integral correctly	3
• Integrates the transformed u-variable expression using standard power rule forms	2
• Correctly transforms the algebraic integrand expression and updates domain bounds limits	1

Sample answer:

$$I = \int_0^{\frac{\pi}{4}} \frac{\sin^2 x}{\cos^4 x}\, dx$$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{\sin^2 x}{\cos^2 x}\right) \cdot \frac{1}{\cos^2 x}\, dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^2 x \sec^2 x\, dx$$

Let $u = \tan x \implies du = \sec^2 x\, dx$

$$x = 0 \implies u = 0, \quad x = \frac{\pi}{4} \implies u = 1$$

$$I = \int_0^1 u^2\, du$$

$$= \left[\frac{u^3}{3}\right]_0^1$$

$$= \frac{1}{3}$$

Question 11 (d)

Criteria	Marks
• Correctly completes the inductive step using double-angle cosine formulas	4
• Proves the inductive statement assuming the result holds true for $n = k$	3
• Establishes a valid structural expression for $n = k + 1$ using nested radicals	2
• Successfully verifies the base case statement for $n = 1$	1

Sample answer:

Let $P(n)$ be: $\underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}_{n \text{ radicals}} = 2\cos\left(\frac{\pi}{2^{n+1}}\right)$

Base Case: $n = 1$

$$\text{LHS} = \sqrt{2}$$

$$\text{RHS} = 2\cos\left(\frac{\pi}{4}\right)$$

$$= \sqrt{2} = \text{LHS}$$

Inductive Step: Assume true for $n = k$

$$\underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}_{k \text{ radicals}} = 2\cos\left(\frac{\pi}{2^{k+1}}\right)$$

Prove true for $n = k + 1$

$$\text{LHS} = \sqrt{2+2\cos\left(\frac{\pi}{2^{k+1}}\right)}$$

$$= \sqrt{4\cos^2\left(\frac{\pi}{2^{k+2}}\right)}$$

$$= 2\cos\left(\frac{\pi}{2^{k+2}}\right) = \text{RHS}$$

Question 11 (e)

Criteria	Marks
• Evaluates the correct total number of non-adjacent alignment arrangements	3
• Calculates correct selection slot allocations utilizing combinations methods	2
• Correctly determines the baseline arrangement configurations of blue and green balls	1

Sample answer:

$$\text{Base arrangements of 4 blue and 4 green balls} = \frac{8!}{4!\,4!}$$

$$= 70$$

$$\text{Ways to place 4 identical red balls in 9 slots} = \binom{9}{4}$$

$$= 126$$

$$\text{Total distinct ways} = 70 \times 126$$

$$= 8820$$

Question 12 (a) (i)

Criteria	Marks
• Correctly derives the transformed variable derivative expression relationship	2
• Employs the product or quotient rule operation properly to separate derivative steps	1

Sample answer:

$$P = vt$$

$$\frac{dP}{dt} = v + t\frac{dv}{dt}$$

$$v + t\frac{dv}{dt} = \frac{vt}{t} + (vt)^2 t$$

$$v + t\frac{dv}{dt} = v + v^2t^3$$

$$t\frac{dv}{dt} = v^2t^3$$

$$\frac{dv}{dt} = v^2t^2$$

Question 12 (a) (ii)

Criteria	Marks
• Solves for $P(t)$ and provides the correct population evaluation value at $t = 2$	2
• Solves the separated v-variable integral and incorporates given initial values conditions	1

Sample answer:

$$\int v^{-2}\,dv = \int t^2\,dt$$

$$-\frac{1}{v} = \frac{t^3}{3} + C$$

At $t = 1$, $P = 200 \implies v = 200$

$$-\frac{1}{200} = \frac{1}{3} + C$$

$$C = -\frac{203}{600}$$

$$-\frac{1}{v} = \frac{t^3}{3} - \frac{203}{600}$$

$$-\frac{1}{v} = \frac{200t^3 - 203}{600}$$

$$v = \frac{600}{203 - 200t^3}$$

$$P(t) = \frac{600t}{203 - 200t^3}$$

At $t = 2$: $$P(2) = \frac{600(2)}{203 - 200(8)}$$

$$= \frac{1200}{-1397}$$

$$\approx -0.86$$

Question 12 (b)

Criteria	Marks
• Proves perpendicularity successfully by demonstrating $\vec{AM}\cdot\vec{BC}=0$ via dot product properties	4
• Incorporates the midpoint vector definition $\mathbf{m}=\frac{\mathbf{b}+\mathbf{c}}{2}$ into equations	3
• Links structural conditions to show $\mathbf{b}\cdot\mathbf{b}=\mathbf{c}\cdot\mathbf{c}$ from the isosceles properties status	2
• Formulates vector expressions tracking lines $\vec{AM}$ and $\vec{BC}$ components parameters	1

Sample answer:

Since $OB=OC \implies |\mathbf{b}|=|\mathbf{c}| \implies \mathbf{b}\cdot\mathbf{b}=\mathbf{c}\cdot\mathbf{c}$

$$\mathbf{a}\cdot(\mathbf{b}-\mathbf{c})=0 \implies \mathbf{a}\cdot\mathbf{b}=\mathbf{a}\cdot\mathbf{c}$$

$$\vec{AM}=\frac{\mathbf{b}+\mathbf{c}}{2}-\mathbf{a}$$

$$vec BC=\mathbf{c}-\mathbf{b}$$

$$\begin{aligned}\vec{AM}\cdot\vec{BC}&=\left(\frac{\mathbf{b}+\mathbf{c}}{2}-\mathbf{a}\right)\cdot(\mathbf{c}-\mathbf{b})\\&=\frac{1}{2}(\mathbf{c}\cdot\mathbf{c}-\mathbf{b}\cdot\mathbf{b})-(\mathbf{a}\cdot\mathbf{c}-\mathbf{a}\cdot\mathbf{b})\\&=\frac{1}{2}(0)-(0)\\&=0\\&\implies \vec{AM}\perp\vec{BC}\end{aligned}$$

Question 12 (c)

Criteria	Marks
• Completes the structural algebraic proof showing $n \approx 40$ flawlessly	4
• Introduces the specified approximation equation to simplify cubic equations parameters	3
• Employs standard deviation formula $\sigma_{\hat{P}} = \sqrt{\frac{p(1-p)}{n}}$	2
• Maps normal distribution threshold score constants correctly from data metrics	1

Sample answer:

$P(Z > z_0) = 0.025 \implies z_0 = 1.96 \approx 2$

$$\frac{0.1}{\sqrt{\frac{p(1-p)}{n}}} = 2$$

$$\frac{0.01n}{p(1-p)} = 4$$

$$\frac{0.01n}{\left(1 - \frac{4}{n}\right)\left(\frac{4}{n}\right)} = 4$$

$$\frac{0.01n^3}{4(n-4)} = 4$$

$$0.01n^3 = 16(n-4)$$

$$n^3 - 1600n + 6400 = 0$$

$$(n-4)(n^2 - 1600) \approx 0$$

$$n^2 - 1600 \approx 0$$

$$n \approx 40$$

Question 12 (d)

Criteria	Marks
• Provides all correct solution	3
• Solves transformed algebraic modular boundary roots successfully via factorizations	2
• Employs proper t-formulae variable transformations expressions correctly	1

Sample answer:

Let $\theta = 2x \implies \cos\theta - \sin\theta < -1$

Let $t = \tan\left(\frac{\theta}{2}\right) = \tan x$

$$\frac{1-t^2}{1+t^2} - \frac{2t}{1+t^2} < -1$$

$$1 - t^2 - 2t < -1 - t^2$$

$$1 - 2t < -1$$

$$2 < 2t$$

$$t > 1$$

$$\tan x > 1$$

$$x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right) \cup \left(\frac{5\pi}{4}, \frac{3\pi}{2}\right)$$

Question 13 (a)

Criteria	Marks
• Successfully derives the given summation expression results using definitive integration tools	3
• Integrates both polynomial expansion layouts component paths appropriately across limits	2
• Recalls standard binomial statement transformations templates accurately	1

Sample answer:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

$$\int_0^1 (1+x)^n \, dx = \int_0^1 \left(\sum_{k=0}^{n} \binom{n}{k} x^k \right) dx$$

$$\left[\frac{(1+x)^{n+1}}{n+1} \right]_0^1 = \sum_{k=0}^{n} \binom{n}{k} \left[\frac{x^{k+1}}{k+1} \right]_0^1$$

$$\frac{2^{n+1}-1}{n+1} = \sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k}$$

Question 13 (b) (i)

Criteria	Marks
• Derives the accurate volume formula layout relation cleanly using double angle identities	2
• Expresses the function x in terms of y and forms the initial volume of revolution template	1

Sample answer:

$$y = \sin^{-1} x$$

$$x = \sin y$$

$$V = \pi \int_0^h x^2 \, dy$$

$$= \pi \int_0^h \sin^2 y \, dy$$

$$= \pi \int_0^h \left(\frac{1 - \cos 2y}{2} \right) dy$$

$$= \frac{\pi}{2} \left[y - \frac{\sin 2y}{2} \right]_0^h$$

$$= \frac{\pi}{2} \left(h - \frac{1}{2} \sin 2h \right)$$

Question 13 (b) (ii)

Criteria	Marks
• Evaluates the correct numeric value for the constant rate k	2
• Employs the calculus chain rule system relating rate derivatives variables accurately	1

Sample answer:

$$\frac{dV}{dh} = \frac{\pi}{2}(1 - \cos 2h)$$

At $h = \frac{\pi}{4}$:
$$\frac{dV}{dh} = \frac{\pi}{2}\left(1 - \cos\frac{\pi}{2}\right)$$
$$= \frac{\pi}{2}$$

$$k = \frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$
$$= \frac{\pi}{2} \times \frac{2}{\pi}$$
$$= 1 \text{ m}^3/\text{min}$$

Question 13 (c) (i)

Criteria	Marks
• Correctly derives the target separation statement formula representation	2
• Equates the horizontal displacement equations mapping mutual target collision coordinates	1

Sample answer:

$$x_1(T) = u_1 T \cos\alpha$$

$$x_2(T) = d - u_2 T \cos\beta$$

$$x_1(T) = x_2(T)$$

$$u_1 T \cos\alpha = d - u_2 T \cos\beta$$

$$d = T(u_1 \cos\alpha + u_2 \cos\beta)$$

Question 13 (c) (ii)

Criteria	Marks
• Evaluates the correct exact launch angle parameters values settings	4
• Sets up valid optimization derivative tracking operations for peak elevation paths	3
• Substitutes given numerical indicators safely to isolate flight timelines constraints	2
• Expresses the horizontal relation link parameters correctly using problem constants	1

Sample answer:

$$40 = T(20\cos\alpha + 20\cos\alpha)$$

$$40 = 40T\cos\alpha$$

$$T = \frac{1}{\cos\alpha}$$

$$\begin{aligned} y_c &= u_1 T\sin\alpha - \frac{1}{2}gT^2 \\ &= 20\left(\frac{1}{\cos\alpha}\right)\sin\alpha - 5\left(\frac{1}{\cos\alpha}\right)^2 \\ &= 20\tan\alpha - 5\sec^2\alpha \\ &= 20\tan\alpha - 5(1+\tan^2\alpha) \\ &= -5\tan^2\alpha + 20\tan\alpha - 5 \end{aligned}$$

Let $w = \tan\alpha \implies \dfrac{dy_c}{dw} = -10w + 20$

$$0 = -10w + 20$$

$$w = 2$$

$$\tan\alpha = 2$$

$$\alpha = \tan^{-1}(2) \approx 63.43°$$

Question 13 (d)

Criteria	Marks
• Evaluates the correct exact value for parameter constant a	3
• Integrates functions expressions correctly to match polynomial coefficients scales	2
• Formulates the correct functional inverse equation $f^{-1}(x)$ manually	1

Sample answer:

$$y = \frac{x+a}{x-a}$$

$$yx - ya = x + a$$

$$x(y-1) = a(y+1)$$

$$x = a\left(\frac{y+1}{y-1}\right)$$

$$f^{-1}(x) = a\left(\frac{x+1}{x-1}\right)$$

$$f^{-1}\left(\frac{x}{a^2}\right) = a\left(\frac{\frac{x}{a^2}+1}{\frac{x}{a^2}-1}\right)$$

$$= a\left(\frac{x+a^2}{x-a^2}\right)$$

$$\frac{x+a}{x-a} = \frac{ax+a^3}{x-a^2}$$

Comparing terms and coefficients: $a = 1$

Question 14 (a) (i)

Criteria	Marks
• Derives the given root expression profile cleanly via vector expansions	2
• Expresses the magnitude dot product square statement format appropriately	1

Sample answer:

$$
\begin{aligned}
f(\theta)^2 &= |\mathbf{a}+\mathbf{b}|^2 \\
&= (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) \\
&= \mathbf{a}\cdot\mathbf{a} + 2(\mathbf{a}\cdot\mathbf{b}) + \mathbf{b}\cdot\mathbf{b} \\
&= |\mathbf{a}|^2 + 2|\mathbf{a}||\mathbf{b}|\cos\theta + |\mathbf{b}|^2 \\
&= 2^2 + 2(2)(3)\cos\theta + 3^2 \\
&= 4 + 12\cos\theta + 9 \\
&= 13 + 12\cos\theta
\end{aligned}
$$

$$f(\theta) = \sqrt{13+12\cos\theta}$$

Question 14 (a) (ii)

Criteria	Marks
• Provides both the correct maximum value constant and angle condition setting θ	2
• Restricts cosine values intervals properly using vector bounds inequalities	1

Sample answer:

$$\mathbf{a} \cdot \mathbf{b} = 6\cos\theta$$

$$6\cos\theta \leq 2$$

$$\cos\theta \leq \frac{1}{3}$$

$\therefore$ Maximum when $\cos\theta = \frac{1}{3}$

$$f(\theta)_{\text{max}} = \sqrt{13 + 12\left(\frac{1}{3}\right)}$$

$$= \sqrt{13+4}$$

$$= \sqrt{17}$$

$$\theta = \cos^{-1}\left(\frac{1}{3}\right)$$

Question 14 (b) (i)

Criteria	Marks
• Proves the inequality chain expression bounds successfully using compound angle identities	2
• Establishes a single inverse tangent expression term by applying tangent identities	1

Sample answer:

Let $A = \tan^{-1}(x+1) \implies \tan A = x+1$

Let $B = \tan^{-1}(x) \implies \tan B = x$

$$\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A \tan B}$$

$$= \frac{(x+1)-x}{1+(x+1)x}$$

$$= \frac{1}{1+x^2+x}$$

$$\tan^{-1}(x+1) - \tan^{-1}(x) = \tan^{-1}\left(\frac{1}{1+x^2+x}\right)$$

For $x > 0$: $\quad 1+x^2 < 1+x^2+x < 1+x^2+2x+1 = 1+(x+1)^2$

$$\frac{1}{1+(x+1)^2} < \frac{1}{1+x^2+x} < \frac{1}{1+x^2}$$

Since $\tan^{-1}(t)$ is a strictly increasing function for $t > 0$:

$$\tan^{-1}\left(\frac{1}{1+(x+1)^2}\right) < \tan^{-1}(x+1) - \tan^{-1}(x) < \tan^{-1}\left(\frac{1}{1+x^2}\right)$$

Question 14 (b) (ii)

Criteria	Marks
• Completes the infinite telescopic summation bounding proof perfectly	3
• Employs the lower-bound inequality relation tracking component terms indicators	2
• Formulates a telescopic series structure model expansion scheme successfully	1

Sample answer:

Let $x = n - 1 \implies \dfrac{1}{1+n^2} < \tan^{-1}(n) - \tan^{-1}(n-1)$

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2} < \sum_{n=1}^{\infty} \left[\tan^{-1}(n) - \tan^{-1}(n-1)\right]$$

Telescoping sum to N : $S_N = \tan^{-1}(N) - \tan^{-1}(0)$

$$= \tan^{-1}(N)$$

$$\lim_{N\to\infty} \tan^{-1}(N) = \frac{\pi}{2}$$

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2} < \frac{\pi}{2}$$

Question 14 (c)

Criteria	Marks
• Correctly completes the inductive step from $k \to k+2$ using vector geometry definitions	5
• Proves the inductive step assuming the result holds true for $n = k$	4
• Correctly establishes and verifies both baseline parameters conditions for $n = 1$ and $n = 2$	3
• Formulates the geometric relationship $\mathbf{p}_n = \overrightarrow{A_{n-1}A_n}$ in terms of midpoints	2
• Successfully calculates initial position coordinates configurations vectors	1

Sample answer:

Let $P(n)$ be the statement: $\mathbf{p}_n = \dfrac{1}{2^{n-1}}\left[\mathbf{a}\cos\left(\dfrac{(n-1)\pi}{2}\right) + \mathbf{b}\left(\dfrac{1-(-1)^{n-1}}{4}\right)\right]$

Base Case 1: For $n = 1$

$$\text{LHS} = \overrightarrow{A_0A_1} = \mathbf{a}$$

$$\text{RHS} = \frac{1}{2^0}\left[\mathbf{a}\cos(0) + \mathbf{b}\left(\frac{1-1}{4}\right)\right] = \mathbf{a} = \text{LHS} \qquad \Longrightarrow P(1) \text{ is true.}$$

Base Case 2: For $n = 2$

A_2 is the midpoint of $A_0B \implies \overrightarrow{OA_2} = \dfrac{1}{2}\mathbf{b}$

$$\mathbf{p}_2 = \overrightarrow{A_1A_2} = \overrightarrow{OA_2} - \overrightarrow{OA_1} = \frac{1}{2}\mathbf{b} - \mathbf{a}$$

$$\text{RHS of } P(2) = \frac{1}{2^1}\left[\mathbf{a}\cos\left(\frac{\pi}{2}\right) + \mathbf{b}\left(\frac{1-(-1)^1}{4}\right)\right]$$

$$= \frac{1}{2}\left[\mathbf{a}(0) + \mathbf{b}\left(\frac{2}{4}\right)\right] = \frac{1}{4}\mathbf{b}$$

Inductive Step: Assume $P(k)$ is true. Prove that $P(k+2)$ is true.

A_{k+2} is the midpoint of $A_kB \implies \vec{OA}_{k+2} = \frac{1}{2}\left(\vec{OA}_k + \mathbf{b}\right)$

A_{k+1} is the midpoint of $A_{k-1}B \implies \vec{OA}_{k+1} = \frac{1}{2}\left(\vec{OA}_{k-1} + \mathbf{b}\right)$

$$\begin{aligned}
\mathbf{p}_{k+2} &= \vec{A_{k+1}A}_{k+2} = \vec{OA}_{k+2} - \vec{OA}_{k+1} \\
&= \frac{1}{2}\left(\vec{OA}_k + \mathbf{b}\right) - \frac{1}{2}\left(\vec{OA}_{k-1} + \mathbf{b}\right) \\
&= \frac{1}{2}\left(\vec{OA}_k - \vec{OA}_{k-1}\right) = \frac{1}{2}\vec{A_{k-1}A}_k = \frac{1}{2}\mathbf{p}_k
\end{aligned}$$

Substituting the assumption from $P(k)$:

$$\begin{aligned}
\mathbf{p}_{k+2} &= \frac{1}{2} \cdot \left(\frac{1}{2^{k-1}} \left[\mathbf{a} \cos\left(\frac{(k-1)\pi}{2}\right) + \mathbf{b}\left(\frac{1-(-1)^{k-1}}{4}\right) \right] \right) \\
&= \frac{1}{2^k} \left[\mathbf{a} \cos\left(\frac{(k-1)\pi}{2}\right) + \mathbf{b}\left(\frac{1-(-1)^{k-1}}{4}\right) \right]
\end{aligned}$$

Using trigonometric shifts properties: $\cos\left(\frac{(k+1)\pi}{2}\right) = -\cos\left(\frac{(k-1)\pi}{2}\right)$

and alternating sign tracks parity balances: $(-1)^{k+1} = -(-1)^{k-1}$

$$\mathbf{p}_{k+2} = \frac{1}{2^{(k+2)-1}} \left[\mathbf{a} \cos\left(\frac{((k+2)-1)\pi}{2}\right) + \mathbf{b}\left(\frac{1-(-1)^{(k+2)-1}}{4}\right) \right] = \text{RHS of } P(k+2)$$

Since $P(1)$ and $P(2)$ are true, and $P(k) \implies P(k+2)$, the statement holds for all integers $n \geq 1$ by mathematical induction.